

Infimum of the exponential volume growth and the bottom of the essential spectrum of the Laplacian

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Abstract

The purpose of this paper is to point out that ‘supremum’ in two inequalities of Brooks [B1] and [B3] should be replaced with ‘infimum’.

1 Introduction

The Laplace-Beltrami operator Δ on a noncompact complete Riemannian manifold M is essentially self-adjoint on $C_0^\infty(M)$ and its self-adjoint extension to $L^2(M)$ has been studied by several authors from various points of view. Especially, the bottom $\min \sigma_{\text{ess}}(-\Delta)$ of the essential spectrum of the Laplacian is simply characterized by the variational formula

$$\min \sigma_{\text{ess}}(-\Delta) = \lim_K \inf_{0 \neq f \in C_0^\infty(M-K)} \frac{\int_M |\nabla f|^2 dv_M}{\int_M |f|^2 dv_M},$$

where K runs over an increasing set of compact subdomains of M such that $\cup K = M$. This bottom $\min \sigma_{\text{ess}}(-\Delta)$ was studied by Donnelly, Brooks, and Sunada and so on.

Donnelly proved in [D] among others that $\min \sigma_{\text{ess}}(-\Delta) \leq (n-1)^2 k/4$ when the Ricci curvature of M is bounded from below by the constant $-(n-1)k$, where $n = \dim M$ and $k \leq 0$. Later, Brooks generalized this Donnelly’s theorem when the volume of M is infinite:

Theorem 1.1 (Brooks [B1]). *Let M be a complete Riemannian manifold and set*

$$\mu = \limsup_{r \rightarrow \infty} \frac{\log \text{vol}(B_{x_0}(r))}{r}.$$

If the volume of M is infinite, then we have

$$\min \sigma_{\text{ess}}(-\Delta) \leq \frac{\mu^2}{4}.$$

When the volume of M is finite, Brooks [B3] also proved

Theorem 1.2 (Brooks [B3]). *Let M be a complete Riemannian manifold and suppose that the volume of M is finite. Let us set*

$$\mu_f = \limsup_{r \rightarrow \infty} \left[-\frac{\log(\text{vol}(M) - \text{vol}(B_{x_0}(r)))}{r} \right].$$

Then we have

$$\min \sigma_{\text{ess}}(-\Delta) \leq \frac{\mu_f^2}{4}.$$

The purpose of this paper is to improve this two Brooks' theorems. Theorem 1.1 is improved as follows, that is, 'lim sup' should be replaced with 'lim inf':

Theorem 1.3. *Let M be a complete Riemannian manifold and set*

$$\mu_{\text{inf}} = \liminf_{r \rightarrow \infty} \frac{\log \text{vol}(B_{x_0}(r))}{r}. \quad (1)$$

If the volume of M is infinite, then we have

$$\min \sigma_{\text{ess}}(-\Delta) \leq \frac{\mu_{\text{inf}}^2}{4}.$$

Theorem 1.2 is also improved as follows, that is, 'lim sup' should be replaced with 'lim inf':

Theorem 1.4. *Let M be a complete Riemannian manifold and suppose that the volume of M is finite. Let us set*

$$\mu_{f,\text{inf}} = \liminf_{r \rightarrow \infty} \left[-\frac{\log(\text{vol}(M) - \text{vol}(B_{x_0}(r)))}{r} \right].$$

Then we have

$$\min \sigma_{\text{ess}}(-\Delta) \leq \frac{\mu_{f,\text{inf}}^2}{4}.$$

2 Proof of theorems

Theorem 1.3 and 1.4 will follow the following

Theorem 2.1. *Let M be a complete Riemannian manifold and K be a compact (possibly empty) set of M . We denote $\lambda_0(M - K) = \min \sigma(-\Delta_{D,M-K})$, where $\Delta_{D,M-K}$ stands for the Dirichlet Laplacian of $M - K$ and $\sigma(-\Delta_{D,M-K})$ is the spectrum of $-\Delta_{D,M-K}$. We suppose that there exist an increasing sequence $\{K_i\}$ of compact subsets of M and positive constants α and d such that*

$$\begin{aligned} B_d(K) &\subset K_1, \quad \cup_{i=1}^{\infty} K_i = M, \\ \lim_{i \rightarrow \infty} \int_{B_d(\partial K_i)} e^{-2\alpha r(x)} dv_M(x) &= 0. \end{aligned} \quad (2)$$

Here, for $A \subset M$, $B_d(A)$ represents the d -neighborhood of A and $r(x)$ is the distance function from a fixed point of M . Then, if

$$0 < \alpha < \sqrt{\lambda_0(M - K)}, \quad (3)$$

we have

$$\int_{M-K} e^{2\alpha r(x)} dv_M(x) < \infty. \quad (4)$$

Theorem 2.1 is proved quite the same way as in Brooks [B1, Theorem 2], and hence, we shall omit its proof.

Proof of Theorem 1.3. We shall set

$$\begin{aligned} K_t &= \{x \in M \mid \text{dist}(x, K) \leq t\} \quad \text{for } t \geq 1, \\ V(r) &= \text{vol}(B_r(K)). \end{aligned}$$

Then, the assumption (1) implies that

$$\liminf_{r \rightarrow \infty} \frac{\log V(r)}{r} = \mu_{\inf}. \quad (5)$$

We remark that if $2\alpha > \mu_{\inf}$, then there exists an increasing sequence r_i of positive numbers such that $\lim_{i \rightarrow \infty} r_i = \infty$ and

$$\lim_{i \rightarrow \infty} V(r_i + 1)e^{-2\alpha r_i} = 0. \quad (6)$$

Indeed, if there is no such sequence (6), there exist positive real numbers ε and r_0 such that for all $r \geq r_0$

$$V(r + 1)e^{-2\alpha r} \geq \varepsilon.$$

Hence, we have

$$\frac{\log V(r)}{r} \geq 2\alpha + \frac{\varepsilon e^{-2\alpha}}{r} \quad \text{for all } r \geq r_0 + 1. \quad (7)$$

But (7) contradicts our assumptions (5) and $2\alpha > \mu_{\inf}$.

From (6), we obtain

$$\begin{aligned} \int_{B_1(\partial K_{r_i})} e^{-2\alpha r(x)} dv_M(x) &\leq (V(r_i + 1) - V(r_i - 1)) e^{-2\alpha r_i} \\ &\leq V(r_i + 1)e^{-2\alpha r_i} \rightarrow 0 \quad (i \rightarrow \infty). \end{aligned}$$

Thus, when we set $K_i = K_{r_i}$ in Theorem 2.1, the assumption (2) holds. Therefore, we now conclude from Theorem 2.1 that $2\alpha > \mu_{\inf}$ and (3) imply that (4). But it is impossible, since $M - K$ has infinite volume. Hence there is no such α , and we have $2\sqrt{\lambda_0(M - K)} \leq \mu_{\inf}$, that is, $\lambda_0(M - K) \leq \mu_{\inf}^2/4$. Taking the limit over arbitrary large K , we get $\min \sigma_{\text{ess}}(-\Delta) \leq \mu_{\inf}^2/4$. We have thus proved Theorem 1.3.

Proof of Theorem 1.4. For simplicity, we shall set $V(r) := \text{vol}(M) - \text{vol}(B_{x_0}(r))$, and take a compact subset K of M . Then, $\lim_{r \rightarrow \infty} V(r) = 0$, since M has finite volume. We remark that if $2\alpha > \mu_{f,\inf}$, then there exists a sequence of positive numbers $\{r_i\}_{i=1}^{\infty}$ such that

$$\begin{aligned} B_{x_0}(r_1) &\supset K, \\ r_{i+1} &\geq r_i + 1, \end{aligned} \quad (8)$$

$$V(r_{i+1})e^{2\alpha r_i} \leq 1, \quad (9)$$

$$2\alpha > \frac{-1}{r_i} \log V(r_i). \quad (10)$$

The inequality (9) comes from the fact that $\lim_{r \rightarrow \infty} V(r) = 0$, and (10) implies

$$e^{2\alpha r_i} V(r_i) \geq 1 \quad \text{for all } i \geq 1. \quad (11)$$

Therefore, for any integer $k \geq 1$, we have

$$\begin{aligned} & \int_{M-K} e^{2\alpha r(x)} dv_M(x) \\ & \geq \sum_{i=1}^k \{\text{vol}(B_{x_0}(r_{i+1})) - \text{vol}(B_{x_0}(r_i))\} e^{2\alpha r_i} \\ & = \sum_{i=1}^k \{V(r_i) - V(r_{i+1})\} e^{2\alpha r_i} \\ & = V(r_1) e^{2\alpha r_1} + \sum_{i=2}^k V(r_i) e^{2\alpha r_i} \{1 - e^{2\alpha(r_{i-1}-r_i)}\} - V(r_{k+1}) e^{2\alpha r_k} \\ & \geq V(r_1) e^{2\alpha r_1} + (k-1) \{1 - e^{-2\alpha}\} - 1. \end{aligned}$$

In the last line, we have used (11), (8), and (9) in turn. Letting $k \rightarrow \infty$, we get

$$\int_{M-K} e^{2\alpha r(x)} dv_M(x) = \infty \quad (12)$$

Now, when we set $K_i = \{x \in M \mid \text{dist}(x, K) \leq r_i\}$, K_i and α satisfy

$$\cup_{i=1}^{\infty} K_i = M, \quad \text{and} \quad \lim_{i \rightarrow \infty} \int_{B_1(\partial K_i)} e^{-2\alpha r(x)} dv_M(x) = 0,$$

since the volume of M is finite. Hence, if $\alpha > \mu_{f,\text{inf}}/2$ satisfies $0 < \alpha < \sqrt{\lambda_0(M-K)}$, Theorem 2.1 implies

$$\int_{M-K} e^{2\alpha r(x)} dv_M(x) < \infty.$$

But this contradicts (12). Therefore, there is no such α , and hence, we get $\mu_{f,\text{inf}}^2/4 \geq \lambda_0(M-K)$. Taking the limit over arbitrary large K , we get $\min \sigma_{\text{ess}}(-\Delta) \leq \mu_{f,\text{inf}}^2/4$. We have thus proved Theorem 1.4.

3 Example and remark

In this section, we shall consider the sharpness of our theorems. We will begin our discussion by considering a rotationally symmetric manifold $(\mathbf{R}^n, g = dr^2 + f(r)^2 g_{S^{n-1}(1)})$. Here, we take an increase sequence of positive numbers $0 < a_1 < a_2 < \dots \rightarrow \infty$ and define

$$f(r) = \begin{cases} 1, & \text{if } a_{4k+1} \leq r \leq a_{4k+2}, \\ e^r, & \text{if } a_{4k+3} \leq r \leq a_{4k+4}, \end{cases}$$

where $k = 0, 1, 2, \dots$. We also assume that f is monotone on the intervals $[a_{4k+2}, a_{4k+3}]$ and $[a_{4k+4}, a_{4k+5}]$ for $k = 0, 1, 2, \dots$. Then, if we choose ‘exponential-intervals’ $[a_3, a_4], [a_7, a_8], \dots$, successively large, then the function $r^{-1} \log \text{vol}(B_0(r))$

will oscillate between the values 0 and $n - 1$, where $B_0(r)$ is the ball centered at the origin 0 with radius r . Thus we will have

$$\begin{aligned}\mu &= \limsup_{r \rightarrow \infty} \frac{\log \text{vol}(B_0(r))}{r} = n - 1, \\ \mu_{\inf} &= \liminf_{r \rightarrow \infty} \frac{\log \text{vol}(B_0(r))}{r} = 0, \\ \min \sigma_{\text{ess}}(-\Delta) &= 0.\end{aligned}$$

This example shows that Theorem 1.3 is indeed sharper than Theorem 1.1. We can also construct an example with similar nature which shows that Theorem 1.4 is indeed sharper than Theorem 1.2.

On the other hand, Theorem 1.3 may fail to be sharp. Indeed, as is pointed out by Brooks [B1], there exists a solvable group G with exponential growth. One such group is given in Milnor [M]. Then if N be any compact manifold with $\pi_1(N, x_0) = G$ and M is the Riemannian universal cover of N , a lemma of Milnor says that $\mu_{\inf} > 0$, while a Brooks' theorem in [B2] (see also Sunada [S]) implies that $\min \sigma_{\text{ess}}(-\Delta) = 0$, since G is amenable. The reason why this gap occurs is that the distance spheres need not to be the most efficient candidates for the isoperimetric inequalities of M . Now let us recall the following Følner-Brooks theorem:

Theorem 3.1 (Følner-Brooks [B2]). *Let N be an n -dimensional compact Riemannian manifold, and M its Riemannian universal cover of N . Then $\pi_1(N, x_0)$ is amenable if and only if, for every (possibly disconnected) fundamental set of M , and for every $\varepsilon > 0$, there exists a finite subset E of $\pi_1(N, x_0)$ such that*

$$H = \bigcup_{g \in E} g \cdot F$$

satisfies the isoperimetric inequality:

$$\frac{\text{vol}_{n-1}(\partial H)}{\text{vol}_n(H)} < \varepsilon.$$

From the proof of Theorem 3.1 and a lemma of Milnor [M], we see that for any $r_0 \geq \text{diam}(N)$, there exist a sequence of finite subsets E_i of $\pi_1(N, x_0)$ such that

$$H_i = \bigcup_{g \in E_i} g \cdot B_{x_0}(r_0)$$

satisfies

$$\lim_{i \rightarrow \infty} \frac{\text{vol}_{n-1}(\partial H_i)}{\text{vol}_n(H_i)} = 0.$$

Thus, in this case, we see that the distance spheres in Theorem 1.3 should be replaced with the *finite union H_i of distance spheres transformed by covering transformations E_i* .

Our main concern above has been the bottom of the essential spectrum. But as for the essential spectrum itself, we note that there is a simple criterion:

Theorem 3.2 ([K1],[K2]). *Let M be a complete Riemannian manifold and assume that there exists an open subset U of M with C^∞ compact boundary ∂U such that the outward exponential map $\exp_{\partial U}^\perp : N^+(\partial U) \rightarrow M - U$ induces a diffeomorphism. We set $r(x) = \text{dist}(x, U)$ for $x \in M - U$. If $\Delta r \rightarrow c$ as $r \rightarrow \infty$ for a constant $c \in \mathbf{R}$, then $[c^2/4, \infty) \subset \sigma_{\text{ess}}(-\Delta)$. Moreover, when U is relatively compact, the equality holds: $\sigma_{\text{ess}}(-\Delta) = [c^2/4, \infty)$.*

References

- [B1] R. Brooks, A relation between growth and the spectrum of the Laplacian, Math. Z., **178** (1981), 501–508.
- [B2] R. Brooks, The fundamental group and the spectrum of the Laplacian, Comment. Math. Helv., **56** (1981), 581–598.
- [B3] R. Brooks, On the spectrum of non-compact manifolds with finite volume, Math. Z., **187** (1984), 425–432.
- [D] H. Donnelly, On the essential spectrum of a complete Riemannian manifold, Topology, **20** (1981), 1–14.
- [K1] H. Kumura, On the essential spectrum of the Laplacian on complete manifolds, J. Math. Soc. Japan, **49** (1997), 1–14.
- [K2] H. Kumura, On the essential spectrum of the Laplacian and vague convergence of the curvatures at infinity, Communications in Partial Differential Equations, **30** (2005), 1555–1565.
- [M] J. Milnor, A note on curvature and fundamental group, J. Differential Geom., **2** (1968), 1–7.
- [S] T. Sunada, Riemannian coverings and isospectral manifolds, Ann. of Math., **121** (1985), 169–186.

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After submitting this article to this preprint server, Professor Yusuke Higuchi let me know that Theorem 1.3 and 1.4 are already known by him (see the following paper). So, I want to delete this article. However, the organizers of this preprint server refuse to delete this article. Hence, I should announce that this article will be unpublished.

Yu.HIGUCHI, A remark on exponential growth and the spectrum of the Laplacian, Kodai Math. J. **24** (2001), 42–47.